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Passage of relativistic electrons through a plasma

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Abstract. A general expression, in terms of thermodynamic Green functions, for the electron–electron direct scattering rate within a hot plasma is evaluated in the case where one of the electrons is relativistic. The result is used to calculate the rate of energy loss of a relativistic electron to a plasma in a wide, but specified, range of temperature and density.

1. Introduction

Scattering and stopping of charged particles passing through a plasma can be treated by expressing the scattering rate for the process in terms of a two-particle thermodynamic Green function. The non-relativistic case has been examined in some detail by Larkin (1960). The formalism and method are more generally applicable and are used here to calculate the rate of energy loss of a relativistic electron to a hot plasma.

2. The scattering rate

If a fast electron has a momentum greater than the average thermal momentum of the plasma electrons, its main energy loss process is the direct term of the electron–electron scattering amplitude, represented by the diagram (figure 1).

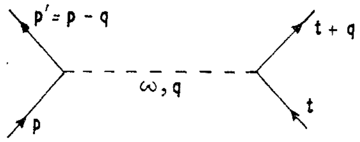

Figure 1. Electron–electron direct scatter.

The squared amplitude can be expressed (Alekseev 1961) in terms of a two-particle thermodynamic Green function with identical pairs of arguments, which can, in turn, be written as a function of the polarization operator for the plasma system. Thus the transition probability for energy–momentum loss $k = (\omega, \mathbf{q})$ is obtained as

$$W(k) = \frac{e^2}{4\pi^3(\mathbf{q}^2-\omega^2)^2\{\exp(-\beta\omega)-1\}} \mathscr{I} \left\{ \frac{T_{\mu\nu}\Pi_{\mu\nu}(\mathbf{q}, -i\omega+\delta)}{1-(\mathbf{q}^2-\omega^2)^{-1}\Pi_{\lambda\lambda}(\mathbf{q}, -i\omega+\delta)} \right\}_{\delta\to 0+} \tag{1}$$

where e^2 is the fine structure constant, $\beta^{-1} = k_{\mathrm{B}}T$, T is the plasma temperature, m the electron mass and $\hbar = c = 1$.

The tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \frac{-1}{8\epsilon(p)\epsilon(p')} \mathrm{Tr}\{\gamma^\nu(\not{p}' + im)\gamma^\mu(\not{p} + im)\} \qquad \not{p} = \gamma p$$

$$= \frac{1}{2\epsilon(p)\epsilon(p')}\{\delta_{\mu\nu}(pp' + m^2) - p_\mu p_\nu' - p_\mu' p_\nu\}. \tag{2}$$

With errors of magnitude e^2 and $e^4\beta^3\rho$, where ρ is the plasma density, the polarization operator Π can be approximated by its zero-order term

$$\Pi_{\mu\nu}(k) = \frac{e^2}{(2\pi)^3\beta} \sum_{n=-\infty}^{\infty} \int dt\, \mathrm{Tr}\{\gamma_\mu G^0(t)\gamma_\nu G^0(t+k)\} \tag{3}$$

where G^0 is the free-electron Green function:

$$G^0(p) = (i\not{p}+m)^{-1} \qquad p_4 = (2n+1)\pi\beta^{-1}+i\mu$$

μ being the chemical potential of the system. In terms of the above quantities the fast electron's rate of energy loss is simply

$$\frac{dE}{dt} = \int \{\epsilon(p)-\epsilon(p-k)\}W(k)\,d\mathbf{q}. \tag{4}$$

Using the tensor and gauge properties of Π (Akhiezer and Peletminskii 1960), we can express

$$\frac{(\mathbf{q}^2-\omega^2)^{-1}\Pi_{\mu\nu}}{1-(\mathbf{q}^2-\omega^2)^{-1}\Pi_{\lambda\lambda}}$$

in the coordinate system defined by $k = (\omega, q, 0, 0)$ and $p_3 = 0$, in the form

$$\begin{bmatrix} -\dfrac{\omega^2}{q^2}\dfrac{h}{1-(q^2-\omega^2)q^{-2}h} & 0 & 0 & -\dfrac{i\omega}{q}\dfrac{h}{1-(q^2-\omega^2)q^{-2}h} \\ 0 & \dfrac{f}{1-f} & 0 & 0 \\ 0 & 0 & \dfrac{f}{1-f} & 0 \\ -\dfrac{i\omega}{q}\dfrac{h}{1-(q^2-\omega^2)q^{-2}h} & 0 & 0 & \dfrac{h}{1-(q^2-\omega^2)q^{-2}h} \end{bmatrix} \tag{5}$$

where

$$h = \frac{\Pi_{44}}{q^2-\omega^2}$$

$$= \frac{-2e^2}{(2\pi)^3(q^2-\omega^2)}\int d\mathbf{t}\,\frac{n(\epsilon)}{\epsilon}\,\frac{(q^2-\omega^2-2\mathbf{t}\,.\,\mathbf{q})(4\epsilon^2-2\mathbf{t}\,.\,\mathbf{q})+4\epsilon^2\omega^2}{(q^2-\omega^2-2\mathbf{t}\,.\,\mathbf{q})^2-4\epsilon^2\omega^2} \tag{6}$$

$$\epsilon = (\mathbf{t}^2+m^2)^{1/2}, \qquad n(\epsilon) = \exp\{-\beta(\epsilon-\mu)\}$$

and

$$f = \frac{1}{2(q^2-\omega^2)}\left\{\Pi_{\nu\nu}-\frac{q^2-\omega^2}{q^2}\,\Pi_{44}\right\}$$

$$\Pi_{\nu\nu} = \frac{-e^2}{2\pi^3}\int d\mathbf{t}\,\frac{n(\epsilon)}{\epsilon}\,\frac{(q^2-\omega^2-2\mathbf{t}\,.\,\mathbf{q})(2m^2-2\mathbf{t}\,.\,\mathbf{q})-4\epsilon^2\omega^2}{(q^2-\omega^2-2\mathbf{t}\,.\,\mathbf{q})^2-4\epsilon^2\omega^2}. \tag{7}$$

Finally, from their method of construction, the components of (5), as functions of complex ω, are analytic in the upper half-plane (Alekseev 1961).

In the same coordinate system as used above we can write (2), for electrons with $|\mathbf{p}| \gg m$ and $\omega = \mathbf{p}\,.\,\mathbf{q}/|\mathbf{p}| = qx$,

$$T_{\mu\nu} = \begin{bmatrix} \dfrac{\omega^2}{q^2} & \cdot & \cdot & \dfrac{i\omega}{q} \\ \cdot & \tfrac{1}{2}-\dfrac{\tfrac{1}{2}\omega^2}{q^2} & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ \dfrac{i\omega}{q} & \cdot & \cdot & -1 \end{bmatrix}$$

where the unspecified entries do not contribute on account of the form of (5).

In proceeding with the calculation of dE/dt it is convenient to split the q integral into two ranges $q > s$, $q < s$, where s is at present unspecified, and to separate the transverse and longitudinal contributions, that is the 2,2; 3,3 entries of (5) and the others, respectively.

3. Low q longitudinal part

In this case, from (1) and (4),

$$\frac{dE}{dt} = \frac{e^2}{2\pi^2}\int_0^s \frac{dq}{q}\mathscr{I}\left\{\int_{-q}^{q} d\omega \frac{\omega}{1-\exp(-\beta\omega)}\,\frac{(q^2-\omega^2)h}{q^2-(q^2-\omega^2)h}\right\}. \tag{8}$$

In this, the maximum value of ω is q, so that if s is chosen equal to β^{-1} the integrand of (8) is analytic in the upper half ω plane. The integral over ω from $-q$ to q along the real axis is now equal to the integral taken round the upper semicircle centred on the origin and of radius q. On this arc we can approximate h, to terms in $(\beta m)^{-1}$, by

$$\begin{aligned} h &= \frac{-e^2}{4\pi^3}\frac{1}{q^2-\omega^2}\int dt\frac{n(\epsilon)}{\epsilon}\frac{(\mathbf{t}\,.\,\mathbf{q})^2+\epsilon^2 q^2}{(\mathbf{t}\,.\,\mathbf{q})^2-\epsilon^2\omega^2} \\ &= \frac{e^2}{\pi^2}\frac{1}{q^2-\omega^2}\frac{q^2}{\omega^2}\int_0^\infty \frac{t^2}{\epsilon}\exp\{-\beta(\epsilon-\mu)\} \\ &= \frac{e^2 m}{\pi^2\beta}\,e^{\beta\mu}K_1(\beta m)\frac{q^2}{\omega^2}\frac{1}{q^2-\omega^2} = \frac{A^2}{q^2-\omega^2}\frac{q^2}{\omega^2} \end{aligned}$$

where K_1 is a first-order modified Bessel function of the second kind (Watson 1958).

The imaginary part of the integral in (8) is given by half of the sum of the residues of such poles of the integrand as lie on the real axis between $-q$ and $+q$. These poles lie at $\omega = \pm A$, so that the imaginary part of the inner integral in (8) has the values

$$\left.\begin{matrix} 0 & \text{if} \quad q < A \\ \tfrac{1}{2}\pi A^2 & \text{if} \quad q > A \end{matrix}\right\}.$$

We thus have

$$\frac{dE}{dt} = \frac{e^2A^2}{4\pi}\int_A^s \frac{dq}{q} = \frac{e^2A^2}{4\pi}\log\left(\frac{s}{A}\right).$$

Expanding to order $(\beta m)^{-1}$, we have

$$\begin{aligned} e^{\beta\mu} &= \frac{1}{2}\left(\frac{2\pi\beta}{m}\right)^{3/2}\rho\, e^{\beta m}\left(1-\frac{15}{8\beta m}\right) \\ K_1(\beta m) &= \left(\frac{\pi}{2\beta m}\right)^{1/2} e^{-\beta m}\left(1+\frac{3}{8\beta m}\right) \end{aligned}$$

so that to this order

$$\frac{dE}{dt} = \frac{e^2A^2}{4\pi}\log\left(\frac{s}{A}\right) \tag{9}$$

where

$$A^2 = \frac{e^2\rho}{m}\left(1-\frac{3}{2\beta m}\right).$$

4. High q longitudinal part

The function $\Pi_{44}(\omega, \mathbf{q})$ is always of the order of, or less than, $\Pi_{44}(0, 0)$, which from (6) can be shown to be less than $e^2\beta\rho$. Thus in this upper range of q it holds that

$$q^2 > s^2 = \beta^{-2} \gg e^2\beta\rho > \Pi_{44}(0) > \Pi_{44}(k)$$

if the plasma temperature and density are such that

$$e^2\beta^3\rho \ll 1. \tag{10}$$

It is now possible to neglect $(q^2-\omega^2)q^{-2}h$ compared with 1 in the denominators in (5). This gives

$$\frac{dE}{dt} = \frac{e^2}{2\pi^2}\int_s^\infty q^2\,dq\int_{-1}^{1} dx\frac{\omega}{1-\exp(-\beta\omega)}\frac{q^2-\omega^2}{q^4}\mathscr{I}h \tag{11}$$

where h is as defined by (6). If the integrand of (6) is considered as a function of complex t, the imaginary part of h arises from the poles of this integrand. To terms of order $(\beta m)^{-1}$ it has the value

$$\mathscr{I}h = \frac{e^2mU}{2\pi\beta q^3}\frac{1-\exp(-\beta\omega)}{(1-x^2)^{5/2}}\exp\{-\beta(U-\mu)\} \tag{12}$$

with

$$U = [\{\tfrac{1}{2}q+xm(1-x^2)^{-1/2}\}^2+m^2]^{1/2}$$

giving in (11)

$$\frac{dE}{dt} = \frac{e^4m}{4\pi^3\beta}\int_s^\infty \frac{dq}{q^2}\int_{-1}^{1} dx\,x(1-x^2)^{-3/2}U\exp\{-\beta(U-\mu)\}. \tag{13}$$

A change of variable to $v = x(1-x^2)^{-1/2}$ brings the inner integral to

$$e^{\beta\mu}\int_{-\infty}^{\infty} dv\,vU(v)(1+v^2)^{-1/2}\exp\left[-\beta m\left\{\left(\frac{q}{2m}+v\right)^2+1\right\}^{1/2}\right]. \tag{14}$$

This is evaluated approximately by substituting, in the factor multiplying the exponential, the value $v = -q/2m$ at which the exponential is minimal, and then performing the integration over the exponential alone. To order $(\beta m)^{-1}$, (14) is

$$\frac{-2\pi^2\beta\rho q}{m(4m^2+q^2)^{1/2}}\left(1-\frac{3}{2\beta m}\right).$$

The remaining integration in (13) is simply done to give, using the fact that $s = \beta^{-1} \ll m$,

$$\frac{dE}{dt} = \frac{e^4\rho}{4\pi m}\left(1-\frac{3}{2\beta m}\right)\log\left(\frac{4m}{s}\right). \tag{15}$$

5. Transverse part

So far we have considered only the longitudinal losses and have neglected the 2,2; 3,3 components of (5). These transverse parts are negligible in the cases considered by Larkin (1960), but here provide a significant further energy loss. If the q integration is split into two ranges as before, it is found that the poles in the low q region always lie outside the range of integration and so give no contribution to the imaginary part. This means that the transverse part is not screened out in the manner of the longitudinal part. It follows that, analogous to (11), the total transverse loss is given by

$$\frac{dE}{dt} = \frac{e^2}{4\pi^2}\int_0^\infty q^2\,dq\int_{-1}^{1} dx\frac{\omega}{1-\exp(-\beta\omega)}\frac{-1}{q^2}\mathscr{I}f. \tag{16}$$

The imaginary part of f is evaluated just as $\mathscr{I}h$ in (12), but with the appropriate linear combination (7) of $\Pi_{\nu\nu}$ and Π_{44} in place of Π_{44} alone. The result, to terms in $(\beta m)^{-1}$, is

$$\mathscr{I}f = \frac{e^2U}{8\pi\beta mq}\frac{1-\exp(-\beta\omega)}{(1-x^2)^{1/2}}\exp\{-\beta(U-\mu)\}$$

giving for (16)

$$\frac{dE}{dt} = \frac{-e^4}{4(2\pi)^3\beta}\frac{1}{2}\left(\frac{2\pi\beta}{m}\right)^{3/2}\frac{\rho}{m}\left(1-\frac{15}{8\beta m}\right)\int_0^\infty dq\int_{-1}^1 dx\,\frac{xU}{(1-x^2)^{1/2}}\,\mathrm{e}^{-\beta U}.$$

The integral is evaluated just as in (14) to yield

$$\frac{dE}{dt} = \frac{e^4\rho}{8\pi m}\left(1-\frac{3}{2\beta m}\right). \qquad (17)$$

6. Conclusions

The most basic approximation used in the preceding sections is that Π can be approximated by its zero-order term (3), an assumption that is valid if $e^4\beta^3\rho$ is small. The assumption of a Maxwell distribution is valid if $\frac{1}{2}(2\pi\beta/m)^{3/2}\rho$ is small. In the calculation it was assumed for (10) that $e^2\beta^3\rho \ll 1$ and throughout that $(\beta m)^{-1} \ll 1$ and $p/m \gg 1$. Thus, for a relativistic electron (> 10 MeV) passing through a plasma whose temperature T (°K) and density ρ (cm^{-3}) satisfy

$$T < 10^9, \qquad \rho < 10^{24}, \qquad \rho T^{-3} < 10^3$$

the total rate of energy loss to order $(\beta m)^{-1}$, given by (9), (15) and (17), is

$$\frac{dE}{dt} = \frac{e^4\rho}{8\pi m}\left(1-\frac{3}{2\beta m}\right)\left[1+\log\left\{\frac{16m^3}{e^2\rho}\left(1+\frac{3}{2\beta m}\right)\right\}\right].$$

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